Mathematics	(Group-II)	PAPER: II
Time: 2.30 Hours	(SUBJECTIVE TYPE)	Marks: 80

SECTION-I

2. Write short answers to any EIGHT (8) questions: (16)

(i) Express the area A of a circle as a function of its circumference C.

Ans If A and C denote the area and circumference of the circle, respectively and r be the radius of the circle, then

$$A = \pi r^2 \qquad \text{(i)}$$

and $C = 2\pi r$

$$\Rightarrow \quad r = \frac{C}{2\pi} \qquad \text{(ii)}$$

By putting $r = \frac{C}{2\pi}$ in (i), we get

$$A = \pi \left(\frac{C}{2\pi}\right)^2$$

$$= \pi \left(\frac{C^2}{4\pi^2}\right)$$

$$= \frac{1}{4\pi^2} C^2$$

(ii) For the real-valued function $f(x) = \frac{2x + 1}{2x - 1}$, $x > 1$. Find $f^{-1}(x)$.

Ans As $\quad y = f(x)$

So, $\quad y = \frac{2x + 1}{x - 1}$

$$y(x - 1) = 2x + 1$$

$$xy - y = 2x + 1$$

$$xy - 2y = y + 1$$

$$x(y - 2) = y + 1$$

$$\Rightarrow \quad x = \frac{y + 1}{y - 2}$$

Thus, $f^{-1}(y) = \frac{y + 1}{y - 2}$ $\qquad (\because \quad x = f^{-1}(y))$

By replacing y by x, we get
$$f^{-1}(x) = \frac{x+1}{x-2}$$

(III) Evaluate $\displaystyle\lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}}$.

Ans By taking factors of $x - 3$,

$$\lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$\therefore \lim_{x\to 3} \frac{x-3}{\sqrt{x}-\sqrt{3}} = \lim_{x\to 3} \frac{(\sqrt{x}+\sqrt{3})(\sqrt{x}-\sqrt{3})}{\sqrt{x}-\sqrt{3}}$$

$$= \lim_{x\to 3} (\sqrt{x}+\sqrt{3})$$

$$= \sqrt{3}+\sqrt{3}$$

$$= 2\sqrt{3}$$

(iv) Find the domain and range of $g(x) = |x - 3|$.

Ans Given function: $g(x) = |x - 3|$

Domain of $g(x) = (-\infty, \infty)$

Range of $g(x) = [0, \infty]$

(v) If $y = \left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^2$, find $\dfrac{dy}{dx}$.

Ans Let $\quad y = \left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^2$

$$= (\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2\sqrt{x}\cdot\frac{1}{\sqrt{x}}$$

$$y = x + \frac{1}{x} - 2$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x + \frac{1}{x} - 2\right)$$

$$= \frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{1}{x}\right) - \frac{d}{dx}(2)$$

$$= 1 + \frac{x(0) - 1(1)}{x^2} - 0 = 1 - \frac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{x^2 - 1}{x^2}$$

(vi) Find $\frac{dy}{dx}$ if $xy + y^2 = 2$.

Ans

$$xy + y^2 = 2$$

Differentiating both sides, we get

$$\frac{d}{dx}(xy + y^2) = \frac{d}{dx}(2)$$

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$

$$x\frac{dy}{dx} + y\frac{d}{dx}(x) + 2y\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} + y(1) + 2y\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} + 2y\frac{dy}{dx} = -y$$

$$(x + 2y)\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x + 2y}$$

(vii) Differentiate $\sin x$ w.r.t. $\cot x$.

Ans Let $y = \sin x$ and $z = \cot x$

$$\frac{dy}{dx} = \cos x \quad \text{and} \quad \frac{dz}{dx} = -\csc^2 x$$

$$\frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz}$$

$$\frac{dy}{dz} = \cos x - \frac{1}{\csc^2 x} = -\cos x \sin^2 x$$

(viii) Find $\frac{dy}{dx}$ if $y = x^2 \ln \frac{1}{x}$.

Ans

$$y = x^2 \ln\left(\frac{1}{x}\right)$$

Differentiating w.r.t 'x', we have

$$\frac{dy}{dx} = \left(\ln\left(\frac{1}{x}\right)\right)\frac{d}{dx}(x^2) + x^2 \frac{d}{dx}\left(\ln\left(\frac{1}{x}\right)\right)$$

$$= \left(\ln\left(\frac{1}{x}\right)\right)(2x) + x^2 \frac{1}{\frac{1}{x}} \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= 2x \ln\left(\frac{1}{x}\right) + x^2 (x) \cdot \left(\frac{-1}{x^2}\right)$$

$$= 2x \ln\left(\frac{1}{x}\right) - x$$

$$= x\left[2 \ln\left(\frac{1}{x}\right) - 1\right]$$

(ix) Find y_2 if $y = x^2 \cdot e^{-x}$.

Ans Given $y = x^2 \cdot e^{-x}$

Differentiate w.r.t. 'x'

$$y_1 = x^2 \cdot \frac{d}{dx}(e^{-x}) + e^{-x} \cdot \frac{d}{dx}(x^2)$$

$$= x^2 (e^{-x}(-1)) + e^{-x}(2x)$$

$$= -x^2 e^{-x} + 2x e^{-x}$$

$$y_1 = -e^{-x}(x^2 - 2x)$$

Again differentiate w.r.t 'x'

$$y_2 = -\left[e^{-x} \cdot \frac{d}{dx}(x^2 - 2x) + (x^2 - 2x)\frac{d}{dx}(e^{-x})\right]$$

$$= -[e^{-x}(2x - 2) + (x^2 - 2x) \cdot e^{-x}(-1)]$$

$$= -e^{-x}[2x - 2 - x^2 + 2x]$$

$$= -e^{-x}[-x^2 + 4x - 2]$$

$$y_2 = e^{-x}(x^2 - 4x + 2)$$

(x) If $y = \ln(\tan h\, x)$, find $\dfrac{dy}{dx}$.

Ans $y = \ln(\tan h\, x)$

Differentiating w.r.t 'x', we get

$$\frac{dy}{dx} = \frac{d}{dx} \ln(\tan h\, x)$$

$$= \frac{1}{\tan h\, x} \frac{d}{dx}(\tan h\, x)$$

$$= \frac{1}{\tan h\, x} \sec h^2 x$$

$$= \frac{1}{\frac{\sin h\, x}{\cos h\, x}} \cdot \frac{1}{\cos h^2 x}$$

$$= \frac{\cos h\, x}{\sin h\, x} \cdot \frac{1}{\cos h^2 x}$$

$$= \frac{1}{\sin h\, x \cos h\, x}$$

$$= \frac{2}{2 \sin h \, x \cos h \, x}$$

$$= \frac{2}{\sin h \, 2x} = 2 \operatorname{cosec} h \, 2x$$

(xi) Find $\frac{dy}{dx}$ if $y = (x^2 + 5)(x^3 + 7)$.

Ans $y = (x^2 + 5)(x^3 + 7) = x^5 + 5x^3 + 7x^2 + 35$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[x^5 + 5x^3 + 7x^2 + 35]$$

$$= \frac{d}{dx}(x^5) + 5\frac{d}{dx}(x^3) + 7\frac{d}{dx}(x^2) + \frac{d}{dx}(35)$$

$$= 5x^{5-1} + 5 \times 3x^{3-1} + 7 \times 2x^{2-1} + 0$$

$$= 5x^4 + 15x^2 + 14x$$

(xii) Find f′(x) if $f(x) = \sqrt{\ln(e^{2x} + e^{-2x})}$.

Ans Let $u = e^{2x} + e^{-2x}$

Then f(x) becomes

$$f(x) = \sqrt{\ln u} = (\ln u)^{1/2}$$

As $f'(x) = \frac{dy}{dx}$

$$= \frac{dy}{du} \times \frac{du}{dx} \quad \text{(By chain rule)}$$

$$= \frac{d}{du}(\ln u)^{1/2} \times \frac{d}{dx} u$$

$$= \left[\frac{1}{2}(\ln u)^{1/2-1} \cdot \frac{d}{du}(\ln u)\right] \times \frac{d}{dx}(e^{2x} + e^{-2x})$$

$$= \left(\frac{1}{2}(\ln u)^{-1/2} \cdot \frac{1}{u}\right) \cdot (e^{2x} \cdot 2 + e^{-2x}(-2))$$

$$= \left(\frac{1}{2(\ln u)^{1/2}} \cdot \frac{1}{u}\right) \cdot (2)(e^{2x} - e^{-2x})$$

$$= \frac{2}{2} \cdot \left(\frac{1}{\sqrt{\ln u}} \cdot \frac{1}{u}\right)(e^{2x} - e^{-2x})$$

$$= \frac{1}{\sqrt{\ln(e^{2x} + e^{-2x})}} \cdot \frac{1}{(e^{2x} + e^{-2x})}(e^{2x} - e^{-2x})$$

$$= \frac{e^{2x} - 2^{-2x}}{\sqrt{\ln(e^{2x} + e^{-2x})} \, (e^{2x} + e^{-2x})}$$

3. Write short answers to any EIGHT (8) questions: (16)

(i) Use differential to find $\frac{dy}{dx}$ for $xy + x = 4$.

Ans

$$xy + x = 4$$
$$x \, dy + y \, dx + dx = 0$$
$$x \, dy + (y + 1) \, dx = 0$$
$$x \, dy = -(1 + y) \, dx$$
$$\frac{dy}{dx} = -\left(\frac{1 + y}{x}\right)$$

(ii) Evaluate the integral $\int \frac{3x + 2}{\sqrt{x}} \, dx$.

Ans

$$\int \frac{3x + 2}{\sqrt{x}} \, dx = \int \left[\frac{3x}{\sqrt{x}} + \frac{2}{\sqrt{x}}\right] dx$$

$$= 3 \int x^{1/2} \, dx + 2 \int x^{-1/2} \, dx$$

$$= 3 \frac{x^{1/2+1}}{\frac{1}{2} + 1} + 2 \frac{x^{-1/2+1}}{-\frac{1}{2} + 1} + c$$

$$= 3 \cdot \frac{x^{3/2}}{\frac{3}{2}} + 2 \cdot \frac{x^{1/2}}{\frac{1}{2}} + c$$

$$= 2x\sqrt{x} + 4\sqrt{x} + c$$

(iii) Evaluate $\int \frac{x + b}{(x^2 + 2bx + c)^{1/2}} \, dx$.

Ans $\int \frac{x + b}{(x^2 + 2bx + c)^{1/2}} \, dx = \int (x^2 + 2bx + c)^{-1/2} (x + b) \, dx$

If we put $u = x^2 + 2bx + c$, then $du = (2x + 2b) \, dx$

$\Rightarrow \quad \frac{1}{2} du = (x + b) \, dx$

Thus $\int (x^2 + 2bx + c)^{-1/2} (x + b) \, dx = \int u^{-1/2} \cdot \frac{1}{2} du$

$$= \frac{1}{2} \int u^{-1/2} \, dx = \frac{1}{2} \frac{u^{-1/2+1}}{\frac{-1}{2} + 1} + c = \frac{1}{2} \cdot \frac{u^{1/2}}{\frac{1}{2}} + c_1$$

$$= u^{1/2} + c_1 = \sqrt{x^2 + 2bx + c} + c_1$$

(iv) Evaluate $\int e^x (\cos x + \sin x)\, dx$.

Ans Let $f(x) = \sin x$, then $f'(x) = \cos x$,

$$\int e^x (\sin x + \cos x)\, dx = \int e^x (f(x) + f'(x))\, dx$$

$$= \int \frac{d}{dx}(e^x f(x))\, dx$$

$$= e^x f(x) + c$$

$$= e^x \sin x + c$$

(v) Evaluate $\int \frac{(a - b)\, x}{(x - a)(x - b)}\, dx$.

Ans We write $\dfrac{(a - b)\, x}{(x - b)(x - a)} = \dfrac{A}{x - a} + \dfrac{B}{x - b}$

$$= \frac{a}{x - a} + \frac{-b}{x - b} \quad \text{(By partial fractions)}$$

Thus $\int \dfrac{(a - b)\, x}{(x - a)(x - b)}\, dx = \int \left[\dfrac{a}{(x - a)} + \dfrac{-b}{(x - b)} \right] dx$

$$= a \int \frac{1}{x - a}\, dx - b \int \frac{1}{x - b}\, dx$$

$$= a\, \ln|x - a| - b\, \ln|x - b| + c$$

(vi) Evaluate $\displaystyle\int_{-1}^{1} (x^{1/3} + 1)\, dx$.

Ans $\displaystyle\int_{-1}^{1} x^{1/3}\, dx + \int_{-1}^{1} 1\, dx = \left. \dfrac{x^{1/3 + 1}}{\frac{1}{3} + 1} \right|_{-1}^{1} + \left. x \right|_{-1}^{1}$

$$= \frac{3}{4} \left. |x^{4/3} \right|_{-1}^{1} + ((1) - (-1)) = \frac{3}{4}[(1)^{4/3} - (-1)^{4/3}] + 1 + 1$$

$$\left(\because \ (-1)^{1/3} = -1 \right)$$

$$= \frac{3}{4}[(1)^4 - (-1)^4] + 2 = \frac{3}{4}[1 - 1] + 2 = 2$$

(vii) Find the area above the x-axis and under the curve $y = 5 - x^2$ from $x = -1$ to $x = 2$.

Ans The curve cuts the x-axis at $\sqrt{5}$ and $-\sqrt{5}$.
Table of some ordered pairs is given below:

x	$-\sqrt{5}$	-2	-1	0	1	2	$\sqrt{5}$
y	0	1	4	5	4	1	0

Thus required area $= \int\limits_{-1}^{2} (5 - x^2).dx = 5 \int\limits_{-1}^{2} 1 . dx - \int\limits_{-1}^{2} x^2.dx$

$$= 5 \left|x\right|_{-1}^{2} - \left|\frac{x^3}{3}\right|_{-1}^{2}$$

$$= 5(2 - (-1)) - \frac{1}{3}(2^3 - (-1)^3)$$

$$= 5(2 + 1) - \frac{1}{3}(8 - (-1))$$

$$= 5 \times 3 - \frac{1}{3}(8 + 1) = 15 - 3 = 12$$

(viii) Solve differential equation ydx + xdy = 0.

Ans

$$y \, dx + x \, dy = 0$$
$$x \, dy = -y \, dx$$
$$\frac{1}{y} dy = -\frac{1}{x} dx$$

Integrating both sides, we have

$$\int \frac{1}{y} dy = - \int \frac{1}{x} dx$$

$$ln(y) = -ln(x) + ln \, c \quad \Rightarrow \quad ln(x) + ln(y) = ln \, c$$
$$ln(xy) = ln \, c \quad \Rightarrow \quad xy = c$$

(ix) Find mid-point of line segment joining A(–8, 3); B(2, –1).

Ans Mid-point of AB

$$= \left(\frac{-8 + 2}{2}, \frac{3 + (-1)}{2}\right)$$

$$= \left(\frac{-6}{2}, \frac{2}{2}\right) = (-3, 1)$$

(x) Two points 'P' and 'O'' given in xy-coordinate system. Find XY-coordinates of 'P' referred to translated axis O'X and O'Y for P(–2, 6); O'(–3, 2).

Ans As P(x, y) = P(–2, 6) and O'(h, k) = O'(–3, 2).

So, the coordinates of P referred to new coordinate system O'X and O'Y are

$$X = x - h = -2 - (-3) = -2 + 3 = 1$$
$$Y = y - k = 6 - 2 = 4$$

Thus P(X, Y) = P(1, 4)

(xi) Find equation of the line joining $(-5, -3)$ and $(9, -1)$.

Ans Slope of the line through $(-5, -3)$ and $(9, -1) = \dfrac{-1 - (-3)}{9 - (-5)}$

$$= \frac{-1 - (-3)}{9 - (-5)} = \frac{2}{14} = \frac{1}{7}$$

The equation of the required line is

$$y - (-3) = \frac{1}{7}(x - (--5))$$

$\Rightarrow \quad 7(y + 3) = x + 5$

$\qquad 7y + 21 = x + 5$

$\Rightarrow \quad x - 7y - 16 = 0$

(xii) Find equation of vertical line through $(-5, 3)$.

Ans As it is parallel to the y-axis, so the required line is $x = -5$.

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4. Write short answers to any NINE (9) questions: **(18)**

(i) Graph the solution set of given linear inequality in xy-plane: $2x + y \leq 6$.

Ans $x^2 = -16y$

Comparing it with: $x^2 = -4ay$

$$4a = 16 \quad \Rightarrow a = \frac{16}{4} = 4$$

Coordinates of focus: $F(0, -4) = F(0, -4)$
Coordinates of vertex: $V(0, 0)$

(ii) Find the centre and radius of the circle with the given equation $5x^2 + 5y^2 + 14x + 12y - 10 = 0$.

Ans $5x^2 + 5y^2 + 14x + 12y - 10 = 0$ \hfill (i)

Dividing both sides of (i) by '5', we have

$$x^2 + y^2 + \frac{14}{5}x + \frac{12}{5}y - 2 = 0$$

$$\Rightarrow \quad x^2 + y^2 + 2\left(\frac{7}{5}\right)x + 2\left(\frac{6}{5}\right)y - 2 = 0 \hfill (ii)$$

Now comparing (ii) with the general equation of a circle given below

$$x^2 + y^2 + 2gx + 2fy + c = 0 \hfill (iii)$$

We have $g = \dfrac{7}{5}$, $f = \dfrac{6}{5}$, $c = -2$

Thus the centre of the circle (i) is $\left(\dfrac{-7}{5}, \dfrac{-6}{5}\right)$ and

$$r = \sqrt{\left(\frac{7}{5}\right)^2 + \left(\frac{6}{5}\right)^2 - (-2)} \qquad (\because r = \sqrt{g^2 + f^2 - c})$$

or $\quad r = \sqrt{\dfrac{49}{25} + \dfrac{36}{25} + 2} = \sqrt{\dfrac{49 + 36 + 50}{25}}$

$$= \frac{\sqrt{135}}{\sqrt{25}} = \frac{\sqrt{9 \times 15}}{5} = \frac{3 \cdot \sqrt{3}\sqrt{5}}{5} = \frac{3\sqrt{3}}{5}$$

(iii) Find the focus and vertex of the parabola $x^2 = -16y$.

Ans Given, $5x^2 + 5y^2 + 14x + 12y - 10 = 0$

Dividing both sides by '5', we have

$$x^2 + y^2 + \frac{14}{5}x + \frac{12}{5}y - 2 = 0$$

$$x^2 + y^2 + 2\left(\frac{7}{5}\right)x + 2\left(\frac{6}{5}\right)y - 2 = 0 \qquad (1)$$

Also we know,

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Comparing (i) and (ii), we get

$$g = \frac{7}{5}, \quad f = \frac{6}{5}, \quad c = -2$$

Thus the centre of the given circle is

$$(-g, -f) = \left(\frac{-7}{5}, \frac{-6}{5}\right)$$

and radius is

$$r = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{\left(\frac{-7}{5}\right)^2 + \left(\frac{-6}{5}\right)^2 + 2}$$

$$= \sqrt{\frac{49}{25} + \frac{36}{25} + 2}$$

$$= \sqrt{\frac{49 + 36 + 50}{25}}$$

$$= \sqrt{\frac{135}{25}}$$

$$= \frac{\sqrt{9 \times 15}}{5}$$

$$\boxed{r = \frac{3\sqrt{15}}{5}}$$

(iv) Write an equation of parabola with given elements:
Focus $(-3, 1)$; directrix $x - 2y - 3 = 0$.

Ans Let $P(x, y)$ be any point on the parabola.

Let F be the focus $(-3, 1)$.

Then $|PF| = \sqrt{(x - (-3))^2 + (y - 1)^2} = \sqrt{(x + 3)^2 + (y - 1)^2}$

$= \sqrt{x^2 + y^2 + 6x - 2y + 10}$

If the perpendicular drawn from $P(x, y)$ to the directrix

$x - 2y - 3 = 0$ meets it at M, then

$$|PM| = \frac{|x - 2y - 3|}{\sqrt{1 + (-2)^2}} = \frac{|x - 2y - 3|}{\sqrt{5}}$$

By definition of parabola, we have

$|PF| = |PM| \implies |PF|^2 = |PM|^2$, that is,

$$x^2 + y^2 + 6x - 2y + 10 = \frac{(x - 2y - 3)^2}{5}$$

$$= \frac{x^2 + 4y^2 + 9 - 4xy - 6x + 12y}{5}$$

$5x^2 + 5y^2 + 30x - 10y + 50 = x^2 + 4y^2 + 9 - 4xy - 6x + 12y$

$\implies \quad 4x^2 + y^2 + 4xy + 36x - 22y + 41 = 0$

(v) Find an equation of directrices of given hyperbola
$\frac{x^2}{4} - \frac{y^2}{9} = 1$.

Ans
$$\frac{x^2}{4} - \frac{y^2}{9} = 1 \tag{i}$$

From (i) $a^2 = 4 \implies a = 2$ and $b^2 = 9 \implies b = 3$

We know that $c^2 = a^2 + b^2$, that is,

$$c^2 = 4 + 9 = 13 \implies c = \sqrt{13}$$

As, Eccentricity: $e = \frac{c}{a}$, so $e = \frac{\sqrt{13}}{2}$

The directries of the parabola (i)

are $x = \pm \frac{a}{e}$, that is, $x = \pm \frac{2}{\frac{\sqrt{13}}{2}} = \pm \frac{4}{\sqrt{13}}$

(vi) Find the centre and eccentricity of given hyperbola
$\frac{y^2}{16} - \frac{x^2}{9} = 1$.

Ans
$$\frac{y^2}{16} - \frac{x^2}{9} = 1 \tag{i}$$

As the positive term of (i) is $\frac{y^2}{16}$, so the transverse axis of (i) is along the y-axis.

The centre of the hyperbola (i) is (0, 0)

From (i), $a^2 = 16 \Rightarrow a = 4$ and $b^2 = 9 \Rightarrow b = 3$

We know that $c^2 = a^2 + b^2$, that is,

$$c^2 = 16 + 9 = 25 \Rightarrow c = 5$$

Eccentricity $e = \dfrac{c}{a} = \dfrac{5}{4}$

As the transverse axis of (i) is along the y-axis, so foci of (i) are (0, ± 5).

(vii) **Find the unit vector in the same direction as the vector $\underline{v} = [3, -4]$.**

Ans

$$\underline{v} = [3, -4] = 3\underline{i} - 4\underline{j}$$
$$|\underline{v}| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$$

Now, $\quad \underline{u} = \dfrac{1}{|\underline{v}|}\underline{v} = \dfrac{1}{5}[3, -4]$

(u is unit vector in the direction of v)

$$\therefore \quad \underline{u} = \left[\frac{3}{5}, \frac{-4}{5}\right]$$

(viii) **Find the constant a so that the vectors $\underline{v} = \underline{i} - 3\underline{j} + 4\underline{k}$ and $\underline{w} = a\,\underline{i} + 9\,\underline{j} - 12\,\underline{k}$ are parallel.**

Ans

$$\underline{v} = 1\underline{i} - 3\underline{j} + 4\underline{k}$$
$$\underline{w} = a\underline{i} + 9\underline{j} - 12\underline{k}$$

Since $\underline{v}$ and $\underline{w}$ are parallel, therefore,

$$\underline{v} \times \underline{w} = \underline{0}$$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -3 & 4 \\ a & 9 & -12 \end{vmatrix} = 0$$

$$\underline{i}(36 - 36) - \underline{j}(-12 - 4a) + \underline{k}(a + 3a) = 0$$
$$0\underline{i} + (12 + 4a)\underline{j} + (9 + 3a)$$
$$0\underline{i} + 0\underline{j} + 0\underline{k}$$

Equating components of $\underline{j}$ and $\underline{k}$, we have

$$12 + 4a = 0 \quad \text{and} \quad 9 + 3a = 0$$
$$4a = -12a \Rightarrow a = -3$$

(ix) Find a vector of length 2 in the direction opposite that of $\underline{v} = -\underline{i} + \underline{j} + \underline{k}$.

Ans

$$\underline{v} = -\underline{i} + \underline{j} + \underline{k}$$

$$|\underline{v}| = |-\underline{i} + \underline{j} + \underline{k}|$$

$$= \sqrt{(-1) + (-1)^2 + (1)^2} = \sqrt{1 + 1 + 1}$$

$$|\underline{v}| = \sqrt{3}$$

Let $\underline{w}$ be unit vector in the direction of $\underline{w}$, then we have

$$\hat{\underline{w}} = \frac{\underline{v}}{|\underline{v}|} = \frac{-\underline{i} + \underline{j} + \underline{k}}{\sqrt{3}}$$

Thus the required vector of length 2 and direction opposite to $\underline{v}$ is $2(-\underline{w})$ or $-2\underline{w}$, i.e.,

$$-2\left[\frac{-\underline{i} + \underline{j} + \underline{k}}{\sqrt{3}}\right]$$

$$= -2\left[\frac{-\underline{i}}{\sqrt{3}} + \frac{\underline{j}}{\sqrt{3}} + \frac{\underline{k}}{\sqrt{13}}\right]$$

$$= \frac{2}{\sqrt{13}}\underline{i} - \frac{2}{\sqrt{13}}\underline{j} - \frac{2}{\sqrt{13}}\underline{k}$$

(x) Find the cosine of the angle θ between $\underline{u}$ and $\underline{v}$
$\underline{u} = [2, -3, 1]$ and $\underline{v} = [2, 4, 1]$.

Ans

$$\underline{u} = [2, -3, 1] = 2\underline{i} - 3\underline{j} + \underline{k}$$

$$|\underline{u}| = \sqrt{(2)^2 + (-3)^2 + (1)^2}$$

$$\underline{u} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

$$\underline{v} = [2, 4, 1] = 2\underline{i} + 4\underline{j} + \underline{k}$$

$$|\underline{v}| = \sqrt{(2)^2 + (4)^2 + (1)^2}$$

$$|\underline{v}| = \sqrt{4 + 16 + 1} = \sqrt{21}$$

$$|\underline{v}| = \sqrt{21}$$

Also $\underline{u} \cdot \underline{v} = (2\underline{i} - 3\underline{j} + \underline{k}) \cdot (2\underline{i} + 4\underline{j} + \underline{k})$

$$= (2)(2) + (-3)(4) + (1)(1)$$

$$= 4 - 12 + 1 = -7$$

Let θ be angle between $\underline{u}$ and $\underline{v}$. So

$$\cos\theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}||\underline{v}|} = \frac{-7}{\sqrt{14}\sqrt{21}}$$

$$= \frac{-7}{\sqrt{7} \cdot \sqrt{2} \cdot \sqrt{7} \cdot \sqrt{3}}$$

$$= \frac{-7}{7\sqrt{6}} = -\frac{1}{\sqrt{6}}$$

Thus $\cos\theta = \frac{-1}{\sqrt{6}}$

(xi) Compute $\underline{b} \times \underline{a}$. Check your answer by showing that $\underline{b}$ is perpendicular to $\underline{b} \times \underline{a}$:

$$\underline{a} = 2\underline{i} + \underline{j} - \underline{k}; \quad \underline{b} = \underline{i} - \underline{j} + \underline{k}.$$

Ans

$$\underline{b} \times \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \underline{i}(1-1) - \underline{j}(-1-2) + \underline{k}(1+2)$$
$$= 0\underline{i} + 3\underline{j} + 3\underline{k}$$

Also $\underline{b} \cdot (\underline{b} \times \underline{a}) = (\underline{i} - \underline{j} + \underline{k})(0\underline{i} + 3\underline{j} + 3\underline{k})$
$$= 1(0) + (-1)(3) + (1)(3)$$
$$= 0 - 3 + 3 = 0$$

Thus, $\underline{b} \times \underline{a}$ is perpendicular to $\underline{b}$.

(xii) If $\underline{a} + \underline{b} + \underline{c} = 0$, then prove that $\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$.

Ans Since $\underline{a} + \underline{b} + \underline{c} = 0$

Taking cross product with $\underline{a}$, we have

$$\underline{a} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{a} \times 0$$
$$\Rightarrow \quad \underline{a} \times \underline{a} + \underline{a} \times \underline{b} + \underline{a} \times \underline{c} = 0 \quad \text{(Using distribution property)}$$
$$\Rightarrow \quad 0 + \underline{a} \times \underline{b} + \underline{a} \times \underline{c} = 0$$
$$\Rightarrow \quad \underline{a} \times \underline{b} = -(\underline{a} \times \underline{c})$$
$$\underline{a} \times \underline{b} = \underline{c} \times \underline{a} \qquad \qquad \text{(i)}$$

Now taking cross product with $\underline{b}$, we have

$$\underline{b} \times (\underline{a} + \underline{b} + \underline{c}) = \underline{b} \times 0$$
$$\Rightarrow \quad \underline{b} \times \underline{a} + \underline{b} \times \underline{b} + \underline{b} \times \underline{c} = 0 \quad \text{(By distributive property)}$$
$$\Rightarrow \quad -(\underline{a} \times \underline{b}) + 0 + \underline{b} \times \underline{c} = 0$$
$$\Rightarrow \quad \underline{b} \times \underline{c} = \underline{a} \times \underline{b} \qquad \qquad \text{(ii)}$$

From (i) and (ii), we conclude that

$$\underline{a} \times \underline{b} = \underline{b} \times \underline{c} = \underline{c} \times \underline{a}$$

(xiii) Give a force $\underline{F} = 2\underline{i} + \underline{j} - 3\underline{k}$ acting at a point A(1, –2, 1). Find the moment of $\underline{F}$ about the point B(2, 0, –2).

Ans Here $\underline{F} = 2\underline{i} + \underline{j} - 3\underline{k}$

$$\underline{r} = \overrightarrow{BA} = (1-2)\underline{i} + (-2-0)\underline{j} + (1+2)\underline{k}$$

$$\Rightarrow \qquad \underline{r} = -\underline{i} - 2\underline{j} + 3\underline{k}$$

Moment of force about B = $\underline{r} \times \underline{F}$

Now $\underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix}$

$$= \underline{i}(6 - 3) - \underline{j}(3 - 6) + \underline{k}(-1 + 4)$$
$$= 3\underline{i} + 3\underline{j} + 3\underline{k}$$

SECTION-II

NOTE: Attempt any Three (3) questions.

Q.5.(a) Find value of k, if the function (5)

$$f(x) = \begin{cases} \dfrac{\sqrt{2x + 5} - \sqrt{x + 7}}{x - 2} & , \quad x \neq 2 \\ k & , \quad x = 2 \end{cases}$$

is continuous at x = 2.

Ans For Answer see Paper 2018 (Group-I), Q.5.(a).

(b) If $y = \tan(p \tan^{-1} x)$, then show that $(1 + x^2) y_1 - p(1 + y^2) = 0$. (5)

Ans Given

$$y = \tan(p \tan^{-1} x)$$

Differentiate w.r.t x

$$\frac{dy}{dx} = \sec^2(p \tan^{-1} x) \cdot \frac{d}{dx}(p \tan^{-1} x)$$

$$y_1 = \sec^2(p \tan^{-1} x) \cdot p \cdot \frac{1}{1 + x^2}$$

Multiplying both sides by $(1 + x^2)$,

$$(1 + x^2) y_1 = p \sec^2(p \tan^{-1} x)$$
$$= p[1 + \tan^2(p \tan^{-1} x)]$$
$$(1 + x^2) y_1 = p(1 + y^2)$$
$$(1 + x^2) y_1 - p(1 + y^2) = 0$$

Q.6.(a) Evaluate $\displaystyle\int \frac{\sqrt{2}}{\sin x + \cos x}\, dx$. (5)

Ans For Answer see Paper 2019 (Group-I), Q.6.(a).

(b) Find an equation of the line through the intersection of the lines $x - y - 4 = 0$ and $7x + y + 20 = 0$ and parallel to the line $6x + y - 14 = 0$. **(5)**

Ans

$$x - y - 4 = 0 \qquad \text{(i)}$$
$$7x + y + 20 = 0 \qquad \text{(ii)}$$

Adding (i) and (ii), we get

$$8x + 16 = 0 \implies x = -2$$

From (i), $y = x - 4 = -2 - 4 = -6$

Thus, the point of intersection of (i) and (ii) is $(-2, -6)$.

The slope of the required line = the slope of the given line

$$= \frac{-6}{1} = -6$$

So, the equation of the required line is

$$y - (-6) = -6(x - (-2))$$
$$y + 6 = -6x - 12$$
$$6x + y + 18 = 0$$

Q.7.(a) Find the area bounded by the curve $y = x^3 - 4x$ and the x-axis. **(5)**

Ans

$$y = x^3 - 4x = x(x^2 - 4)$$

If $y = 0$, then

$$x(x^2 - 4) = 0$$
$$x = 0 \quad , \qquad x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

So, the given curve cuts the x-axis at $(0, 0)$, $(-2, 0)$, $(2, 0)$.

So, the required area will be:

$$\int_{-2}^{0} y\, dx - \int_{0}^{2} y\, dx = \int_{-2}^{0} (x^3 - 4x)\, dx - \int_{0}^{2} (x^3 - 4x)\, dx$$

$$= \int_{-2}^{0} x^3\, dx - 4\int_{-2}^{0} x\, dx - \int_{0}^{2} x^3\, dx + 4\int_{0}^{2} x\, dx$$

$$= \left.\frac{x^4}{4}\right|_{-2}^{0} - 4\left.\frac{x^2}{2}\right|_{-2}^{0} - \left.\frac{x^4}{4}\right|_{0}^{2} + 4\left.\frac{x^2}{2}\right|_{0}^{2}$$

$$= \left(\frac{(0)^4 - (-2)^4}{4}\right) - 2\{(0)^2 - (-2)^2\} - \left(\frac{(2)^4 - (0)^4}{4}\right) + 2\{(2)^2 - (0)^2\}$$

$$= \frac{-16}{4} - 2(-4) - \frac{16}{4} + 2(4)$$

$$= -4 + 8 - 4 + 8$$
$$= 8$$

(b) Maximize $f(x, y) = 2x + 5y$ subject to the constraints: **(5)**

$$2y - x \leq 8, \ x - y \leq 4, \ x \geq 0, \ y \geq 0$$

Ans.

$$-x + 2y \leq 8 \tag{i}$$
$$x - y \leq 4 \tag{ii}$$

Joining the points $(-8, 0)$ and $(0, 4)$, we get

$$-x + 2y = 8 \tag{iii}$$

of the linear inequality (or constraint) (i). The graph of the inequality (i) is the closed half plane on the origin side of (iii), because

$$(-1) \, 0 + 2(0) = 0 < 8$$
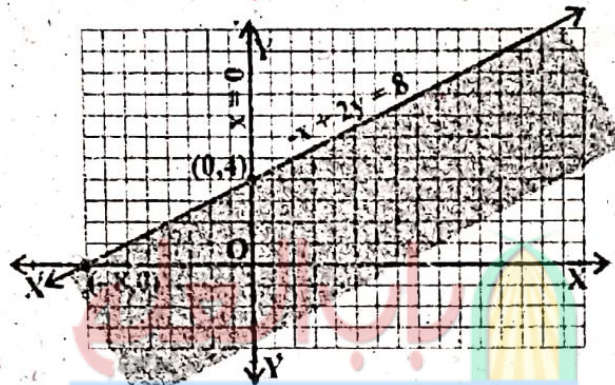
It is partially shown by shading in figure (1).

Fig. 1.

The associated equation

$$x - y = 4 \tag{iv}$$

of the inequality (ii) is drawn by joining the points $(4, 0)$ and $(0, -4)$. The graph of the inequality $x - y \leq 4$ is the closed half plane on the origin side of (iv) because $1.0 - 1.0 = 0 < 4$. In figure (2), it is partially indicated by shading.
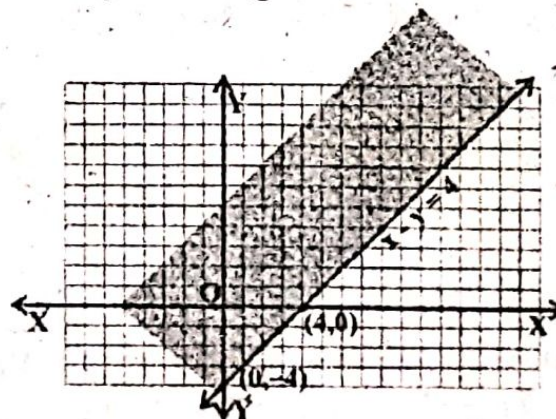

Fig. 2.

The solution region of inequalities (i) and (ii) is the intersection of graphs shown in figures (1) and (2) and it is partially displayed by shading in figure (3).
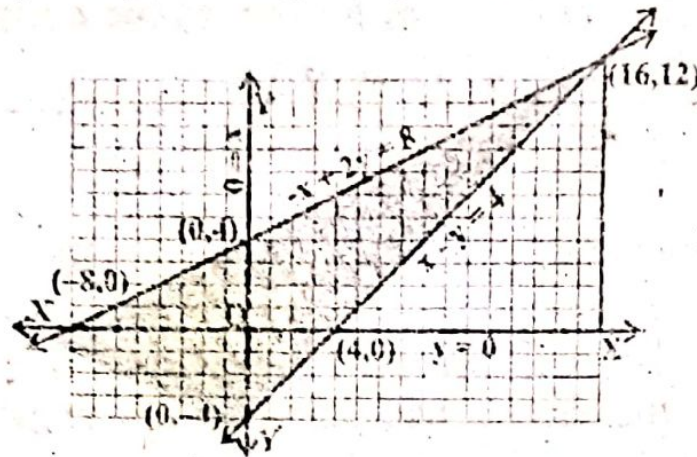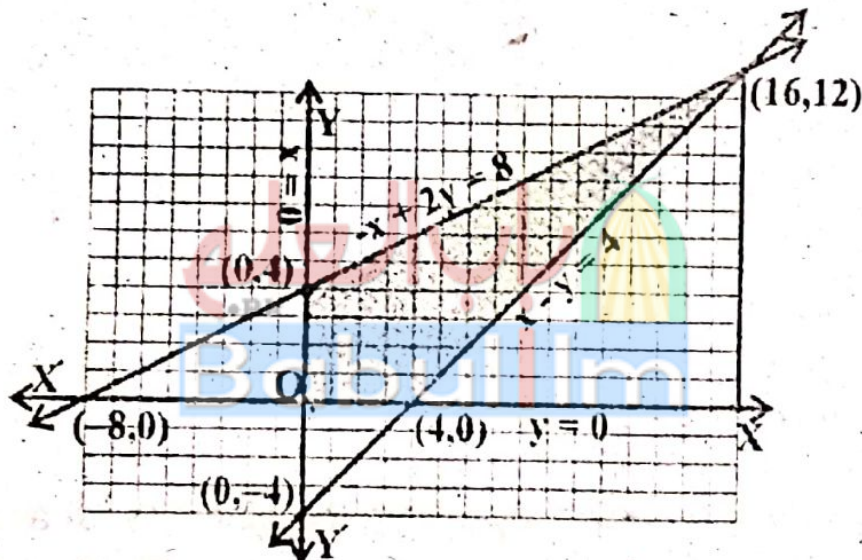

Fig. 3.

Using non-negative constraints, the graph in figure 3 is restricted to the first quadrant. The required feasible region is quadrilateral region shown by shading in figure (4).

Adding (iii) and (iv), we get $y = 12$ and $x = y + 4 = 12 + 4 = 16$, therefore, the point of intersection of (iii) and (iv) is (16, 12).

Thus the corner points of the feasible region are (0, 0), (4, 0), (16, 12) and (0, 4).

Now $f(x, y) = 2x + 5y$

$f(0, 0) = 2(0) + 5(0) = 0$, $f(4, 0) = 2(4) + 5(0) = 8$

$f(16, 12) = 2(16) + 5(12) = 32 + 60 = 92$

$f(0, 4) = 2(0) + 5(4) = 20$

f is maximum at the corner point (16, 12).

Q.8.(a) Write equation of the circle passing through the points A(–7, 7), B(5, –1) and C(10, 0). (5)

Ans Let $x^2 + y^2 + 2gx + 2fy + c = 0$ (i)

Since A(–7, 7), B(5, –1) and C(10, 0) lie on the circle (i), so we have

$$(-7)^2 + (7)^2 + 2g(-7) + 2f(7) + c = 0$$

$$49 + 49 - 14g + 14f + c = 0$$

$$-14g + 14f + c + 98 = 0 \qquad \text{(ii)}$$

$$(5)^2 + (-1)^2 + 2g(5) + 2f(-1) + c = 0$$

$$10g - 2f + c + 26 = 0 \qquad \text{(iii)}$$

$$(10)^2 + (0)^2 + 2g(10) + 2f(0) + c = 0$$

$$20g + c + 100 = 0 \qquad \text{(iv)}$$

Subtracting (ii) from (iii) gives

$$-24g + 16f + 72 = 0 \qquad \Rightarrow \quad 3g - 2f - 9 = 0 \qquad \text{(v)}$$

Subtracting (iii) from (iv) gives $10g + 2f + 74 = 0$ (vi)

Adding (v) and (vi), we get

$$13g + 65 = 0 \qquad \Rightarrow \quad g + 5 = 0$$

$$g = -5$$

Putting g = –5 in (v), we have

$$3(-5) - 2f - 9 = 0$$

$$-2f - 24 = 0$$

$$f + 12 = 0 \qquad \Rightarrow \quad f = -12$$

Putting g = –5 in (iv) gives

$$20(-5) + c + 100 = 0$$

$$-100 + c + 100 = 0 \qquad \Rightarrow \quad c = 0$$

Substituting g = –5, f = –12 and c = 0 in (i), we get

$$x^2 + y^2 + 2(-5)x + 2(-12)y + 0 = 0$$

$$x^2 + y^2 - 10x - 24y = 0$$

(b) **Find a vector of length 5 in the direction opposite that of v = i – 2j + 3k.** (5)

Ans Given vector is:

$$\underline{v} = i - 2j + 3k$$

Then,

$$|\vec{v}| = \sqrt{(1)^2 + (-2)^2 + (3)^2}$$

$$= \sqrt{1 + 4 + 9}$$

$$|\vec{v}| = \sqrt{14}$$

If $\underline{u}$ is a unit vector in the direction of $\vec{v}$, then

$$\underline{u} = \frac{\vec{v}}{|\vec{v}|}$$

$$\underline{u} = \frac{\underline{i} - 2\underline{j} + 3\underline{k}}{\sqrt{14}}$$

$$\underline{u} = \frac{1}{\sqrt{14}} (\underline{i} - 2\underline{j} + 3\underline{k})$$

Now a vector of length 5 in the direction opposite to that of $\vec{v}$ is

$$-5\underline{u} = -5 \left[\frac{1}{\sqrt{14}} (\underline{i} - 2\underline{j} + 3\underline{k}) \right]$$

$$= \frac{-5}{\sqrt{14}} (\underline{i} - 2\underline{j} + 3\underline{k})$$

$$= \frac{-5}{\sqrt{14}} \underline{i} + \frac{10}{\sqrt{14}} \underline{j} - \frac{15}{\sqrt{14}} \underline{k}$$

Q.9.(a) Show that $y = \dfrac{\ln x}{x}$ has maximum value at $x = e$. **(5)**

Ans
$$y = \frac{\ln x}{x} \qquad \text{(i)}$$

By differentiating (i) w.r.t 'x', we have

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{\ln x}{x} \right)$$

$$= \frac{d}{dx} \left[\left(\frac{1}{x} \right) \cdot (\ln x) \right]$$

$$= \frac{1}{x} \frac{d}{dx} (\ln x) + \ln x \frac{d}{dx} \left(\frac{1}{x} \right)$$

$$= \left(\frac{1}{x} \right) \left(\frac{1}{x} \right) + \ln x \left(\frac{-1}{x^2} \right)$$

$$\frac{dy}{dx} = \frac{1}{x^2} - \frac{\ln x}{x^2} \qquad \text{(ii)}$$

If $\dfrac{dy}{dx} = 0$, then

$$\dfrac{1}{x^2} - \dfrac{\ln x}{x^2} = 0$$

$$\dfrac{1 - \ln x}{x^2} = 0$$

$$1 - \ln x = 0$$

$$\Rightarrow \quad \ln x = 1$$

$$\ln x = \ln e \qquad\qquad [\because \ln e = 1]$$

$$x = e$$

Again, differentiating (ii), we have

$$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left[\dfrac{1}{x^2} - \dfrac{\ln x}{x^2}\right]$$

$$= \dfrac{d}{dx}\left(\dfrac{1}{x^2}\right) - \dfrac{d}{dx}\left[(\ln x)\left(\dfrac{1}{x^2}\right)\right]$$

$$= \dfrac{d}{dx}(x^{-2}) - \left[\left(\dfrac{1}{x^2}\right)\dfrac{d}{dx}(\ln x) + \ln x \dfrac{d}{dx}\left(\dfrac{1}{x^2}\right)\right]$$

$$= -2x^3 - \left[\left(\dfrac{1}{x^2}\right)\left(\dfrac{1}{x}\right) + \ln x\,(-2x^{-3})\right]$$

$$= \dfrac{-2}{x^3} - \left[\dfrac{1}{x^3} - \dfrac{2\ln x}{x^3}\right]$$

$$= \dfrac{-2}{x^3} - \dfrac{1}{x^3} + \dfrac{2\ln x}{x^3}$$

$$\dfrac{d^2y}{dx^2} = \dfrac{-3}{x^3} + \dfrac{2\ln x}{x^3}$$

$$\dfrac{d^2y}{dx^2}\bigg|_{x=e} = \dfrac{-3}{e^3} + \dfrac{2\ln e}{e^3}$$

$$= \dfrac{-3}{e^3} + \dfrac{2(1)}{e^3}$$

$$= \dfrac{-3}{e^3} + \dfrac{2}{c^3}$$

$$= \dfrac{-1}{e^3} < 0$$

So, y has maximum value at x = e.

(b) Find focus, vertex and directrix of parabola $x^2 - 4x - 8y + 4 = 0$. **(5)**

Ans.

$$x^2 - 4x - 8y + 4 = 0$$
$$x^2 - 4x + 4 = 9y$$
$$(x - 2)^2 = 8y \qquad \text{(i)}$$

If we shift the origin to $(2, 0)$ by the translation

$$x - 2 = X, \quad y = Y$$

Then (i) becomes

$$X^2 = 8Y \qquad \text{(ii)}$$

Comparing (ii) with $X^2 = 4ay$, we have

$$4a = 8 \quad \Rightarrow \quad a = 2$$

The vertex of (ii) is $(0, 0)$, i.e., $X = 0$ ad $Y = 0$

$$X = 0 \qquad ; \qquad Y = 0$$
$$x - 2 = 0 \qquad ; \qquad y = 0$$
$$x = 2$$

Thus, the focus of (i) is $(2, 0)$. As (ii) opens upwards, so the focus of (ii) is $(0, 2)$, i.e., $X = 0$ and $Y = 2$.

$$X = 0 \qquad ; \qquad Y = 2$$
$$x - 2 = 0 \qquad ; \qquad y = 2$$
$$x = 2$$

Thus, the focus of (i) is $(2, 2)$.
The axis of (ii) is $X = 0$, i.e.,

$$x - 2 = 0 \text{ is the axis of (i)}.$$

Directrix of (ii) is $Y = -2$, i.e.,

$$y = -2 \text{ is the directrix of (i)}.$$