Notes on Fibonacci Numbers

Scribe: Sarah Beaver   Lecturer/Editor: Chris Eppolito

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Today we will study the Fibonacci numbers.

Definition. The *Fibonacci numbers* are defined by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-2} + f_{n-1}$ for all $n \geq 2$.

Proposition. *For all $n \in \mathbb{N}$ we have $\sum_{k=0}^{n} f_k = f_{n+2} - 1$.*

Proof. Let $n \in \mathbb{N}$ be arbitrary and proceed by induction on n to prove $P(n) : \sum_{k=0}^{n} f_k = f_{n+2} - 1$.

Base Case: For $n = 0$ we have $f_{0+2} - 1 = f_2 - 1 = 1 - 1 = 0$ and $\sum_{k=0}^{0} f_k = f_0 = 0$. Thus $\sum_{k=0}^{0} f_k = 0 = f_{0+2} - 1$ as desired.

Inductive Step: Let $j \in \mathbb{N}$ be arbitrary and assume $\sum_{k=0}^{j} f_k = f_{j+2} - 1$. We compute

$$\sum_{k=0}^{j+1} f_k = \sum_{k=0}^{j} f_k + f_{j+1} = (f_{j+2} - 1) + f_{j+1} = f_{j+2} + f_{j+1} - 1 = f_{(j+1)+2} - 1.$$

Therefore $\sum_{k=0}^{j+1} f_k = f_{(j+1)+2} - 1$ and the inductive step holds.

Hence the original statement is true by mathematical induction. □

Proposition. *Let $(a_n)_{n\geq 0}$ be a real sequence. If $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$, then for all $k \in \mathbb{N}$ we have*

$$a_{n+1} = f_{n+1} a_1 + f_n a_0.$$

Proof. We proceed by strong induction. Suppose $(a_n)_{n\geq 0}$ is a sequence of numbers satisfying $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$

Base Case: Note for $n = 1$ and $n = 2$ we have the following, verifying our base cases.

$$\begin{aligned} a_1 &= 1 \cdot a_1 + 0 \cdot a_0 = f_1 a_1 + f_0 a_0 \\ a_2 &= a_1 + a_0 = 1 \cdot a_1 + 1 \cdot a_0 = f_2 a_1 + f_1 a_0 \end{aligned}$$

Inductive Step: Let $n \geq 2$ and suppose $a_k = f_k a_1 + f_{k-a} a_0$ for all $1 \geq k \geq n$. We now compute

$$\begin{aligned} a_{n+1} &= a_n + a_{n-a} \\ &= f_n a_1 + f_{n-1} a_0 + f_{n-1} a_1 + f_{n-2} a_0 \\ &= (f_n + f_{n-1}) a_1 + (f_{n-1} + f_{n-2}) a_0 \\ &= f_{n+1} a_1 + f_n a_0. \end{aligned}$$

Hence $a_{n+1} = f_{n+1} a_1 + f_n a_0$, and the inductive step holds.

We conclude the original statement is true by mathematical induction. □

Corollary. *For all $\alpha, \beta \in \mathbb{N}$ we have $f_{\alpha+\beta+1} = f_{\alpha+1} f_{\beta+1} + f_\alpha f_\beta$.*

Proof. We apply the previous proposition to the shifted Fibonacci sequence, $a_n = f_{\alpha+n}$ for all $n \in \mathbb{N}$. Note $a_n = f_{\alpha+n} = f_{\alpha+(n+1)} + f_{\alpha(n-2)} = a_{n-1} + a_{n-2}$ for all $n \geq 2$. Hence we obtain

$$f_{\alpha+\beta+1} = a_{\beta+1} = f_{\beta+1} a_1 + f_{\beta+1} a_0 = f_{\beta+1} f_{\alpha+1} + f_\beta f_\alpha. \quad \square$$

Corollary. *Let $n, d \in \mathbb{N}$. If $d \mid n$, then $f_d \mid f_n$.*

Proof. Let $d \in \mathbb{N}$ be arbitrary. Note that if $d = 0$, then $d \mid n$ implies $n = 0$. Thus the statement trivially holds ($f_0 \mid f_0$). Otherwise, we assume $d \neq 0$ and we proceed by strong induction on n.

Base Case: If $n = 0$, then trivially $d \mid 0$; moreover, $f_n = f_0 = 0 \cdot f_d$, so $f_d \mid f_n$ in this case, and the statement holds.

Inductive Step: Assume that $d \mid k$ implies $f_d \mid f_k$ for all $0 \geq k \geq n$; further suppose $d \mid n$. By definition of divisibility there is an integer $m \in \mathbb{Z}$ such that $n = dm$. Thus we rewrite $n = dm = (d-1) + d(m-1) + 1$ Applying the previous corollary, we obtain

$$f_n = f_{dm} = f_{(d-1)+d(m-1)+1} = f_{(d-1)+1} f_{d(m-1)+1} + f_{d-1} f_{d(m+1)} = f_d f_{d(m-1)+1} + f_{d(m+1)} f_{d-1}.$$

Now $d(m-1) < dm = n$, so by the induction hypothesis $f_d \mid f_{d(m-1)}$ and thus there is an $l \in \mathbb{Z}$ such that $f_{d(m-1)} = f_d l$ by definition of divisibility. Hence we have

$$f_n = f_d f_{d(m-1)+1} + f_{d(m+1)} f_{d-1} = f_d f_{d(m-1)+1} + (f_d l) f_{d-1} = f_d (f_{d(m-1)+1} + f_{d-1} l).$$

By closure properties of integers, $f_{d(m-1)+1} + f_{d-1} l \in \mathbb{Z}$. Hence $f_d \mid f_n$ and the inductive step holds.

We conclude the original statement is true. ☐