Notes on Subgraphs

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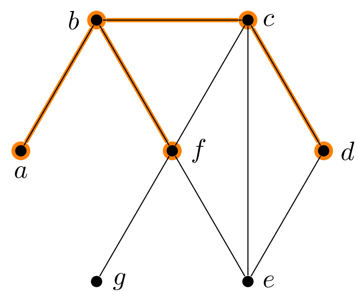
Definition. Let G be a graph. A *subgraph* of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write $H \leq G$ to denote H is a subgraph of G.

Example 1. The graphs G and H defined below satisfy $H \leq G$.

$$G = (\{a, b, c, d, e, f, g\}, \{ab, bc, bf, cd, ce, cf, de, ef, fg\}), \quad \text{and} \quad H = (\{a, b, c, d, f\}, \{ab, bc, bf, cd\})$$

Indeed, we depict G and H below, with H highlighted in orange.

$G =$ 

We worked together as a class to solve the following problem regarding subgraphs.

Problem 1. How many subgraphs does K_n have?

For $n = 2$ we can list all of these fairly easily.

Let $\text{pow}_k(T) \coloneqq \{S \subseteq T : \#S = k\}$ and note $\#\,\text{pow}_k(T) = \binom{\#T}{k}$ by definition of the binomial coefficients.

Solution. The set of subgraphs of K_n is precisely $S = \{(V, E) : V \subseteq [n], E \subseteq \text{pow}_2(V)\}$. We may write

$$\begin{aligned} S &= \{(V, E) : V \subseteq [n], E \subseteq \text{pow}_2(V)\} \\ &= \bigcup_{V \subseteq [n]} \{(V, E) : E \subseteq \text{pow}_2(V)\} \\ &= \bigcup_{k=0}^{n} \bigcup_{V \in \text{pow}_k([n])} \{(V, E) : E \subseteq \text{pow}_2(V)\} \end{aligned}$$

As each of these unions is disjoint, we apply the Sum Principle to obtain the following.

$$\begin{aligned}
\#S &= \# \bigcup_{k=0}^{n} \bigcup_{V \in \mathrm{pow}_k([n])} \{(V, E) : E \subseteq \mathrm{pow}_2(V)\} \\
&= \sum_{k=0}^{n} \# \bigcup_{V \in \mathrm{pow}_k([n])} \{(V, E) : E \subseteq \mathrm{pow}_2(V)\} \\
&= \sum_{k=0}^{n} \sum_{V \in \mathrm{pow}_k([n])} \# \{(V, E) : E \subseteq \mathrm{pow}_2(V)\} \\
&= \sum_{k=0}^{n} \sum_{V \in \mathrm{pow}_k([n])} \# \mathrm{pow}(\mathrm{pow}_2(V)) \\
&= \sum_{k=0}^{n} \sum_{V \in \mathrm{pow}_k([n])} 2^{\# \mathrm{pow}_2(V)} \\
&= \sum_{k=0}^{n} \sum_{V \in \mathrm{pow}_k([n])} 2^{\binom{\# V}{2}} \\
&= \sum_{k=0}^{n} \sum_{V \in \mathrm{pow}_k([n])} 2^{\binom{k}{2}} = \sum_{k=0}^{n} \# \mathrm{pow}_k([n]) 2^{\binom{k}{2}} = \sum_{k=0}^{n} \binom{n}{k} 2^{\binom{k}{2}}
\end{aligned}$$

Hence K_n has precisely $\sum_{k=0}^{n} \binom{n}{k} 2^{\binom{k}{2}}$ subgraphs. □