Notes on the Handshaking Lemma

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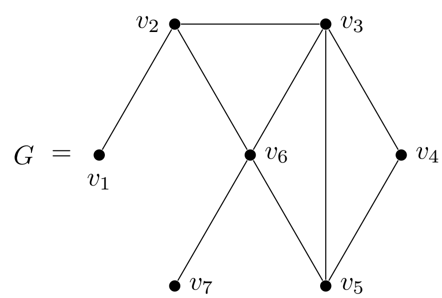
Recall the following example from the previous lecture.

Example 1. Consider the graph G given below; we compute the degrees of the vertices of G.



$\deg_G(v_1) = 1$ $\deg_G(v_2) = 3$ $\deg_G(v_3) = 4$ $\deg_G(v_4) = 2$
$\deg_G(v_5) = 3$ $\deg_G(v_6) = 4$ $\deg_G(v_7) = 1$

The above example has the sum of the degrees is twice the number of edges. This is true in general.

Proposition (Handshaking Lemma)**.** *For every graph G we have*

$$\sum_{v \in V(G)} \deg_G(v) = 2 \cdot \#E(G).$$

Proof. We count the set $S = \{(v, e) \in V(G)xE(G) : v \in e\}$ in two ways.

Note $S = \bigcup_{v \in V(G)} \{(v, e) \in S : v \in e\}$, and this union is disjoint. Hence by the sum principle

$$\#S = \# \bigcup_{v \in V(G)} \{(v, e) \in S : v \in e\} = \sum_{v \in V(G)} \# \{(v, e) : e \in E(G), v \in e\} = \sum v \in V(G) \deg_G(v).$$

On the other hand $S = \bigcup_{e \in E(G)} (v, e) \in S : v \in e$. Noting $\#e = 2$ for all $e \in E(G)$ and that this union is disjoint. Applying the Sum Principle we obtain

$$\#S = \# \bigcup_{e \in E(G)} \{(v, e) \in S : v \in e\} = \sum_{e \in E(G)} \# \{(v, e) : v \in e\} = \sum_{e \in E(G)} 2 = 2 \cdot \#E(G).$$

Hence we have $\sum_{v \subseteq V(G)} \deg_G(v) = \#S = 2 \cdot \#E(G)$ as desired. □