Notes Introducing Simple Graphs

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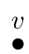
30 March 2020

Today we begin our study of simple graphs, a model for symmetric relationships between objects.

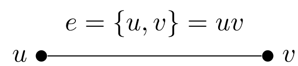
Definition. A *simple graph* is a pair $G = (V(G), E(G))$ of sets with $E(G) \subseteq \{S \subseteq V(G) : \#S = 2\}$. Elements of $V(G)$ are *vertices* and elements of $E(G)$ are *edges*.

For the next several classes we will study simple graphs; we omit the word "simple" for brevity.

We abbreviate $\{u, v\}$ to uv when there is no risk for confusion; note $uv = vu$ with this convention. We often describe graphs via drawings where dots and line segments represent vertices and edges respectively.

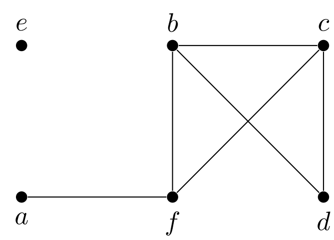


vertex



edge

Example 1. The graph $G = (\{a, b, c, d, e, f\}, \{af, bc, bd, bf, cd, cf\})$ is depicted below.



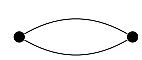
Note that there are many possible drawings of a given graph; it is important to note that the picture is a convenient shorthand for the graph, and the graph itself does not depend on a given drawing.

Let G be a graph with $u, v \in V(G)$ and $e \in E(G)$. We say...

1. u is *adjacent* to v when $uv \in E(G)$.
2. v is *incident* to e when $v \in e$.
3. u and v are the *ends* of e when $e = uv$.

Remark. In a simple graph, we see neither *loops* ("edges" of the form vv) nor *parallel edges* ("double edges" with the same pair of ends). Thus, our pictures won't have the following configurations.

loop



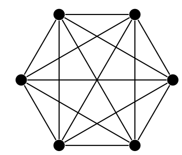
parallel edges

Example 2. The following are some important families of graphs.

1. The *empty graphs* are defined by $\Phi_n = ([n], \emptyset)$ for $n \in \mathbb{N}$.

$\Phi_3 =$

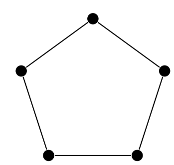
2. The *complete graphs* are defined by $K_n = ([n], \{uv : u, v \in [n], u \neq v\})$ for $n \in \mathbb{N}$.

$K_6 =$ 

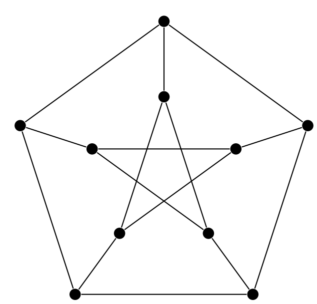
3. The *path graphs* are $P_n = (\{0\} \cup [n], \{\{i-1, i\} : i \in [n]\})$ for $n \in \mathbb{N}$).

$P_4 =$

4. The *cycle graphs* $(for n \in \mathbb{N})$ are $C_n = (\{0\} \cup [n-1], \{\{0, n-1\}\} \cup \{\{i-1, i\} : i \in [n-1]\})$.

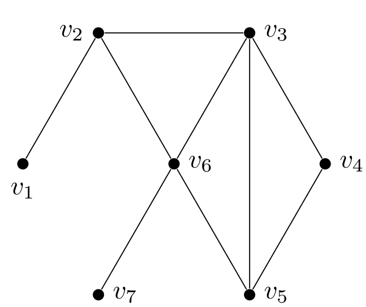
$C_5 =$ 

5. The *Peterson graph* P is depicted below.

$P =$ 

Definition. Let G be a graph and $v \in V(G)$. The *degree* of v in G is $\deg_G(v) = \# \{e \in E(G) : v \text{ is incident to } e\}$.

Example 3. Consider the graph G given below; we compute the degrees of the vertices of G.

$G =$ 

$\deg_G(v_1) = 1$	$\deg_G(v_2) = 3$	$\deg_G(v_3) = 4$	$\deg_G(v_4) = 2$
$\deg_G(v_5) = 3$	$\deg_G(v_6) = 4$	$\deg_G(v_7) = 1$	